Self Gravitating Incompressible Fluid in Two Dimensions

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Abstract

In this paper we develop two models for the steady states and evolution of two dimensional isothermal self gravitating and rotating incompressible gas which are based on the hydrodynamic equations for stratified fluid. The first model is for the steady states of the gas while the second addresses the time evolution of the gas subject to some constraints. These models reduce the initial five partial differential equations that govern this system to two for the steady state model and to three for the time dependent model. Analytical and numerical solutions of the model equations are used to study the structure of the resulting steady and time dependent states of the fluid with some possible astrophysical applications.

1 Introduction

The steady states of self gravitating fluid in three dimensions have been studied by a long list of illustrious mathematical physicists. (For an extensive list of references see [1,2,3]). The motivation for this research was due to the interest in the shape, stability and evolution of celestial bodies and systems[12]. We now know however that many celestial objects such as galaxies and our solar system exhibit (effectively) "two dimensional structure" [4,5,6,7]. Furthermore recent discoveries are leading us to believe that systems similar to our solar system are "abundant" in the galaxy and their existence might be due to the collapse of a two dimensional interstellar cloud under gravitation (this is the so called the "nebular theory") [8,9,10,11,17]. This data leads us to believe that there is a fundamental physical process which we do not understand fully as yet that leads to the formation of planetary systems throughout the galaxy (and beyond).

This background motivates us to investigate in this paper the steady states and time-dependent evolution of a self gravitating and rotating fluid in two dimensions. This problem has been explored by a large number of investigators using elaborate analytic methods and computer simulations which involve, in general, thermodynamic considerations, magnetohydrodynamics modeling and turbulence.(For a complete list of references see [8,9,10,13,16]). While these are important issues we still need, in our opinion, prototype analytic models that are able to capture the evolution of this process and lead to insights about its possible outcomes.

In this paper we attempt to develop such a model using the basic hydrodynamic equations that govern the time-dependent evolution of an isothermal, incompressible, stratified (i.e non constant density) and rotating fluid in two dimensions under gravity [1,2,3]. (The justification for the reduction from three to two dimensions has been discussed by many

authors. A lucid treatment is given in Ref. [12] pp.1-12).Under these assumptions we show that the number of model equations can be reduced from five to a system of two equations for the steady states and three coupled equations for its evolution. The models contain some "parameter functions" which encode information about the asymptotic mass density distribution of the fluid and its momentum.

The steady state model was investigated by us previously.[18,19,20] However in this paper we consider a more general model in which the "gas cloud" is rotating also with uniform angular velocity ω and study the impact of this rotation on the matter distribution in the steady state.

To study the predictions of these models we use both analytical and numerical methods to solve their equations under a variety of conditions. In particular we consider radial solutions to these equations which represent the evolution of an interstellar cloud with isothermal equation of state [10].

It might be argued that the hydrodynamic assumptions we are making in this paper are not realistic from astrophysical point of view. However our main goal is to capture analytically, as far as possible, the nonlinear and time dependent aspects of the processes under consideration. Accordingly our results might be useful to provide some analytic insights and guidelines for more elaborate work on this topic.

The plan of the paper is as follows: In Sec 2 we present the basic hydrodynamic equations and show how one can reduce them to a coupled system of three equations. Sec 3 presents further simplifications of these equations. The first is for the steady states of the model. The second is for the time dependent evolution of the gas cloud under the assumption of constant vorticity. In Sec 4 we present analytical and numerical radial solutions of these equations. We end up Sec 5 with summary and conclusions.

2 Derivation of the Model Equations

Following the standard convention [1,14,15] we model the time dependent non-relativistic flow of an incompressible fluid in two dimensions (x, y) by the hydrodynamic equations of inviscid and incompressible stratified fluid

$$u_x + v_y = 0 \tag{2.1}$$

$$\rho_t + u\rho_x + v\rho_y = 0 \tag{2.2}$$

$$\rho u_t + \rho(uu_x + vu_y) = -p_x - \rho\phi_x + \rho\omega^2 x \tag{2.3}$$

$$\rho v_t + \rho(uv_x + vv_y) = -p_y - \rho\phi_y + \rho\omega^2 y \tag{2.4}$$

$$\nabla^2\phi = 4\pi G\rho \tag{2.5}$$

where subscripts indicate differentiation with respect to the indicated variable, $\mathbf{u} = (u, v)$ is the fluid velocity, ρ is its density, p is the pressure, ϕ is the gravitational field and G is the gravitational constant. The terms $\rho\omega^2 x$, $\rho\omega^2 y$ represent the components of the apparent centrifugal force due to the rotation of the gas cloud with angular velocity ω.

We can nondimensionalize these equations by introducing the following scalings

$$t = \frac{L\tilde{t}}{U_0},\ \ x = L\tilde{x},\ \ y = L\tilde{y},\ \ u = U_0\tilde{u},\ \ v = U_0\tilde{v},\ \ \rho = \rho_0\tilde{\rho},\ \ p = \rho_0 U_0^2\tilde{p},\ \ \phi = U_0^2\tilde{\phi},\ \ \omega = \frac{U_0}{L}\tilde{\omega}. \tag{2.6}$$

where L, U_0, ρ_0 are some characteristic length,velocity and mass density respectively that characterize the problem at hand. Substituting these scalings in eqs. (2.1)-(2.5) and dropping the tildes these equations remain unchanged (but the quantities that appear in these equations become nondimensional) while G is replaced by $\tilde{G} = \frac{G\rho_0 L^2}{U_0^2}$. (Once again we drop the tilde).

In view of eq. (2.1) we can introduce a stream function ψ so that

$$u = \psi_y,\ \ v = -\psi_x\ . \tag{2.7}$$

Using this stream function we can rewrite eq. (2.2) as [13,15]

$$\rho_t + J\{\rho, \psi\} = 0 \tag{2.8}$$

where for any two (smooth) functions f, g

$$J\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{2.9}$$

Using ψ the momentum equations (2.3),(2.4) become

$$\rho(\psi_{yt} + \psi_y \psi_{yx} - \psi_x \psi_{yy}) = -p_x - \rho\phi_x + \rho\omega^2 x \tag{2.10}$$

$$\rho(-\psi_{xt} - \psi_y \psi_{xx} + \psi_x \psi_{xy}) = -p_y - \rho\phi_y + \rho\omega^2 y \tag{2.11}$$

To eliminate p from these equations we differentiate eq. (2.10) and eq. (2.11) with respect to y, x respectively and subtract. This leads to

$$\begin{aligned} &\rho_y(\psi_{yt} + \psi_y \psi_{yx} - \psi_x \psi_{yy}) + \rho(\psi_{yyt} + \psi_y \psi_{yyx} - \psi_x \psi_{yyy}) - \\ &\rho_x(-\psi_{xt} - \psi_y \psi_{xx} + \psi_x \psi_{xy}) - \rho(-\psi_{xxt} - \psi_y \psi_{xxx} + \psi_x \psi_{xxy}) = -J\{\phi, \rho\} + J\{\frac{1}{2}\omega^2 r^2, \rho\} \end{aligned} \tag{2.12}$$

where $r^2 = x^2 + y^2$. The sum of the second and fourth terms in this equation can be rewritten as

$$\rho(\nabla^2 \psi)_t + \rho J\{\nabla^2 \psi, \psi\}. \tag{2.13}$$

To reduce the first and third terms in (2.12) we use (2.8). We obtain

$$\begin{aligned} \rho_y(\psi_{yt} + \psi_y \psi_{yx} - \psi_x \psi_{yy}) - \rho_x(-\psi_{xt} - \psi_y \psi_{xx} + \psi_x \psi_{xy}) = \\ \rho_y(\psi_{yt} + \rho_y \psi_y \psi_{yx} - (\rho_t + \rho_x \psi_y)\psi_{yy} + \rho_x \psi_{xt} + (\psi_x \rho_y - \rho_t)\psi_{xx} - \rho_x \psi_x \psi_{xy} = \\ \rho_y \psi_{yt} + \rho_y \psi_{yt} - \rho_t \nabla^2 \psi + \frac{1}{2} J\{(\psi_x)^2 + (\psi_y)^2, \rho\}. \end{aligned} \tag{2.14}$$

combining the results of (2.13) and (2.14) eq. (2.12) becomes

$$\rho_y\psi_{yt} + \rho_x\psi_{xt} - \rho_t\nabla^2\psi + \rho(\nabla^2\psi)_t + \rho J\{\nabla^2\psi, \psi\} + \frac{1}{2}J\{(\psi_x)^2 + (\psi_y)^2, \rho\} = -J\{\phi, \rho\} + J\{\frac{1}{2}\omega^2 r^2, \rho\} \tag{2.15}$$

Thus we have reduced the original five equations (2.1)-(2.5) to three equations (2.5), (2.8) and (2.15). Although (2.15) is rather cumbersome in general, it can simplified further under some restrictions which are presented in the following section.

3 Simplification of the Model Equations

Equation (2.15) can be simplified further in two cases. The first is when we consider only steady states of the flow and the second is when the flow vorticity is constant.

3.1 A Model for the Steady States

When we consider only steady states of the flow (2.8) implies that $\psi = \psi(\rho)$ and after some algebra [18] (2.15) reduces to

$$H(\rho)^{1/2}\nabla\cdot(H(\rho)^{1/2}\nabla\rho) + \phi - \frac{1}{2}\omega^2 r^2 = S(\rho). \tag{3.1}$$

Where

$$H(\rho) = \rho\psi_\rho^2 \tag{3.2}$$

and $S(\rho)$ is some function of ρ. Thus the equations governing the steady state are (3.1), (2.5) and $H(\rho)$ and $S(\rho)$ are "parameter functions" which determine the nature of the steady state.

3.1.1 The Physical Meaning of the Functions $H(\rho)$, $S(\rho)$

The function $H(\rho)$ is a parameter function which is determined by the momentum (and angular momentum) distribution in the fluid. From a practical point of view the choice of this function determines the structure of the steady state density distribution. The corresponding flow field can be computed then aposteriori (that is after solving for ρ) from the following relations.[18]

$$u = \sqrt{\frac{H(\rho)}{\rho}}\frac{\partial \rho}{\partial y}, \quad v = -\sqrt{\frac{H(\rho)}{\rho}}\frac{\partial \rho}{\partial x}. \tag{3.3}$$

The function $S(\rho)$ that appears in eq. (3.1) can be determined from the asymptotic values of ρ and ϕ on the boundaries of the domain on which eqs (2.5),(3.1) are solved. When these asymptotic values are imposed or known one can evaluate the left hand side of eq. (3.1) on the domain boundaries and re-express it in terms of ρ only to determine $S(\rho)$ on the boundary of the domain. However, the resulting functional relationship of S on ρ must then hold also within the domain itself since S does not depend on x, y directly.

For example if we assume that on an infinite domain $h(\rho) = 1$, $\omega = 0$ and the asymptotic behavior of ρ and ϕ is given by

$$\lim_{r\to\infty} \rho(r) = e^{-\alpha r^2}, \quad \lim_{r\to\infty} \phi(r) = 4\alpha^2 r^2 e^{-\alpha r^2} \tag{3.4}$$

then (asymptotically) (3.1) evaluates to

$$S(\rho) = -4\alpha e^{-\alpha r^2} = -4\alpha\rho \tag{3.5}$$

3.2 A model for the Time Evolution

To begin with we consider the case where the vorticity is zero and then generalize to the case where the flow vorticity is constant.

When the flow vorticity $\nabla \times \mathbf{u}$ is zero then $\nabla^2\psi = 0$ eq. (2.15) becomes

$$\rho_y\psi_{yt} + \rho_x\psi_{xt} + \frac{1}{2}J\{(\psi_x)^2 + (\psi_y)^2, \rho\} = -J\{\phi, \rho\} \tag{3.6}$$

However when the vorticity is zero we can introduce the velocity potential η which satisfies $\eta_x = u$, $\eta_y = v$. Replacing ψ by η in (3.6) we obtain

$$J\{\eta_t + \frac{1}{2}[(\eta_x)^2 + (\eta_y)^2] + \phi, \rho\} = 0 \tag{3.7}$$

Hence

$$\eta_t + \frac{1}{2}[(\eta_x)^2 + (\eta_y)^2] + \phi = S(\rho) \tag{3.8}$$

The equations of the flow in this case are

$$\rho_t + \eta_x\rho_x + \eta_y\rho_y = 0 \tag{3.9}$$

(which replaces (2.8)), (3.8) and (2.5).

To generalize this reduction to the case where $\nabla^2\psi = a$ (where a is any constant) we define

$$v_1 = \psi_y, \;\; v_2 = -\psi_x + ax$$

Therefore

$$(v_1)_y - (v_2)_x = 0,$$

which implies that there exists a function η so that

$$\eta_x = v_1, \;\; \eta_y = v_2.$$

Hence

$$\eta_x = \psi_y, \;\; \eta_y = -\psi_x + ax \tag{3.10}$$

Using these relations to substitute η for ψ in (3.12) leads to

$$\rho_y \eta_{xt} - \rho_x (\eta_y - ax)_t + \left[-a\rho_t + \frac{1}{2} J\{(\eta_y - ax)^2 + (\eta_x)^2, \rho\} \right] = -J\{\phi, \rho\}. \tag{3.11}$$

Therefore

$$J\{\eta_t, \rho\} - a\rho_t + \frac{1}{2} J\{(\eta_y - ax)^2 + (\eta_x)^2, \rho\} = -J\{\phi, \rho\}. \tag{3.12}$$

Hence

$$-a\rho_t + J\{\eta_t + \frac{1}{2}[(\eta_y - ax)^2 + (\eta_x)^2] + \phi, \rho\} = 0. \tag{3.13}$$

Using (2.8) we have

$$-aJ\{\psi, \rho\} + J\{\eta_t + \frac{1}{2}[(\eta_y - ax)^2 + (\eta_x)^2] + \phi, \rho\} \tag{3.14}$$

It follows then that

$$-a\psi + \eta_t + \frac{1}{2}[(\eta_y - ax)^2 + (\eta_x)^2] + \phi = S(\rho). \tag{3.15}$$

If $a \neq 0$, ψ can be eliminated from this equation if we differentiate with respect to y and use (3.10) to obtain

$$-a\eta_x + \left[\eta_t + \frac{1}{2}[(\eta_y - ax)^2 + (\eta_x)^2] + \phi \right]_y = S(\rho)_y \tag{3.16}$$

4 Radial Solutions for the Steady State Model

When we consider the special case where in polar coordinates $\rho = \rho(r)$ and $\phi = \phi(r)$ the system (3.1) and (2.5) with $H(\rho) = 1$ reduces to

$$\rho'' = -\frac{\rho'}{r} + S(\rho) - \phi + \frac{1}{2}\omega^2 r^2 \tag{4.1}$$

$$\phi'' = -\frac{1}{r}\phi' + c\rho, \quad c = 4\pi G \tag{4.2}$$

To solve this system of equations we let $S(\rho) = \alpha\rho$, solve (4.1) for ϕ and substitute the result in (4.2). This leads to the following fourth order equation for ρ

$$\rho'''' + \frac{2}{r}\rho''' - \left(\alpha + \frac{1}{r^2}\right)\rho'' + \left(\frac{1}{r^3} - \frac{1}{r}\right)\rho' + c\rho = 2\omega^2. \tag{4.3}$$

The general solution of this equation is

$$\rho = \frac{2\omega^2}{c} + C_1 J_0(a_1 r) + C_2 J_0(b_1 r) + C_3 Y_0(a_1 r) + C_4 Y_0(b_1 r) \tag{4.4}$$

where J_0 and Y_0 are Bessel functions of the first and second kind of order 0 and

$$a_1 = \frac{1}{2}\sqrt{-2\alpha + 2\sqrt{\alpha^2 - 4c + \alpha^2}},\ \ b_1 = \frac{1}{2}\sqrt{-2\alpha - 2\sqrt{\alpha^2 - 4c + \alpha^2}}$$

Assuming no singularity at the origin we set $C_3 = C_4 = 0$. To assess the impact of the rotation term on the steady state we solved this system for C_1, C_2 on a circular disk using the boundary conditions $\rho(0) = 1$ and $\rho(8) = 0$ with $c = 1$, $\alpha = -19.4$. The results of these computations for different values of ω are plotted in Fig. 1. In this figure we see that the separation between the density peaks become more pronounced as ω increases. This might interpreted as leading to the creation of protoplanets around the central core.

A strong dependence on ω is shown in Fig. 2 which has the same parameters as Fig. 1 except that the boundary conditions on ρ are: $\rho(0) = 0.35$ and $\rho(8) = 0.25$. This figure illustrate clearly the effect that rotation can have on the pattern of density fluctuations within the cloud. Furthermore in this figure the magnitude of the density fluctuations reverses itself as ω becomes larger viz. the higher density peaks are placed at larger values of r. (Which is reminiscent of the situation in the solar system)

5 Radial Solutions for the Time Evolution Model

The system (2.5),(3.8) and (3.9) can be simplified further if we use polar coordinates and assume that ρ, η, ϕ are functions of r and t only. We obtain,

$$\begin{aligned} \rho_t + \eta_r \rho_r &= 0, \\ \phi_{rr} + \frac{1}{r}\phi_r - c\rho &= 0, \\ \eta_t + \frac{1}{2}(\eta_r)^2 + \phi &= S(\rho). \end{aligned} \tag{5.5}$$

where $c = 4\pi G$.

5.1 Steady States

When we consider a steady state solutions of (5.5) then $\rho_t = 0$ and $\eta_t = 0$. If follows from the first equation in (5.5) that either ρ_r or η_r must be zero. In the first case ρ is constant and we can let $\rho = 1$ without loss of generality. When η_r is zero we must have $\phi = S(\rho)$ and the second equation in (5.5) becomes

$$S'(\rho)\left[\rho_{rr} + \frac{1}{r}\rho_r\right] + S''(\rho)(\rho_r)^2 - c\rho = 0 \tag{5.6}$$

where primes denote differentiation with respect to ρ.

We consider these two cases separately.

A. Steady state with $\rho = 1$

Since $\rho = 1$ the function $S(\rho)$ is a constant and the general solution for ϕ is

$$\phi = \frac{c}{4}r^2 + C_1 \ln r + C_2. \tag{5.7}$$

where C_1, C_2 are arbitrary constants. The equation for η becomes

$$\frac{1}{2}(\eta_r)^2 = S - \frac{c}{4}r^2 - C_1 \ln(r) - C_2. \tag{5.8}$$

(S can be absorbed in C_2 but we leave it in this form as these two constants have different physical meaning). If we let $C_1 = 0$ to avoid the singularity at the origin (5.8) yields

$$\eta = \pm \left\{ \frac{1}{4} r \sqrt{8S - 8C_2 - 2cr^2} + \frac{(S - C_2)\sqrt{2}}{\sqrt{c}} \arctan \left[\frac{\sqrt{2c}\, r}{\sqrt{8S - 8C_2 - 2cr^2}} \right] \right\} + C_3 \quad (5.9)$$

B. Steady states with $\eta = 1$

In this case the solution of (5.6) depends on the nature of the function $S(\rho)$. In general this equation has to be solved numerically. However we present here analytical solutions of this equation for two special cases.

1. $S(\rho) = \alpha\rho$ where α is a constant. The solution to (5.6) in this case is

$$\rho = C_4 J_0 \left(\sqrt{-\frac{c}{\alpha}}\, r \right) + C_5 Y_0 \left(\sqrt{-\frac{c}{\alpha}}\, r \right) \quad (5.10)$$

   It follows then that the nature of the steady state is determined by the ratio $\frac{c}{\alpha}$. A sample of the resulting ρ profiles is presented in $Fig.\,3$. To obtain this figure we considered a pinched disk with $\rho(0.01) = 1$, $\rho'(0.01) = -10$ and $c = \alpha$. The resulting steady state has an increase in the material density towards the circumference of the disk. Similar graphs were obtained numerically for $S(\rho) = \alpha\rho^n$, $n = 2, 3$.

2. $S(\rho) = \alpha \ln(\rho)$

   In this case we have

$$\rho = \frac{1}{2C_1 cr^2 \cos(\theta)^2} \quad (5.11)$$

   where

$$\theta = \frac{1}{2\sqrt{C_1 \alpha}} (\ln r - C_2)$$

   Substituting $C_1 = C_2 = c = \alpha = 1$ we obtain $Fig.\,4$ which might be interpreted as representing a binary system.

5.2 Perturbations from the steady state $\rho = 1$

We consider in this section a disk of radius 1 with a steady state $\rho_0 = 1$ and $S(\rho) = 0$. Letting $\phi(1) = 0$ and using (5.7), (5.9) (with $C_1 = 0$) this yields the following equations for the steady state

$$\phi_0(r) = \frac{c}{4}(r^2 - 1)$$
$$\eta_0 = \frac{\sqrt{2c}}{4}\left[\arcsin(r) + r\sqrt{(1 - r^2)}\right]$$

For a perturbation from this state, viz.

$$\rho(t, r) = \rho_0 + \epsilon\rho_1(t, r), \quad \phi(t, r) = \phi_0 + \epsilon\phi_1(t, r), \quad \eta(t, r) = \eta_0 + \epsilon\eta_1(t, r) \tag{5.12}$$

we obtain to first order in ϵ the following system of equations:

$$\begin{aligned} (\rho_1)_t + \frac{\sqrt{2c(1 - r^2)}}{2}(\rho_1)_r &= 0, \\ (\phi_1)_{rr} + \frac{1}{r}(\phi_1)_r - c\rho_1 &= 0, \\ (\eta_1)_t + \frac{\sqrt{2c(1 - r^2)}}{2}(\eta_1)_r + \phi_1 &= 0. \end{aligned} \tag{5.13}$$

The equation for ρ_1 in (5.13) can be solved analytically. Its general solution is

$$\rho_1 = F\left(\sqrt{\frac{2}{c}}\arcsin(r) - t\right) \tag{5.14}$$

where F is any smooth function of its variable which has to be adjusted to the initial conditions of the perturbation. The second equation in (5.13) is a (reduced) Poisson equation and its general solution can be expressed by quadratures

$$\phi_1 = \int \frac{c\{\int rF\left(\sqrt{\frac{2}{c}}\arcsin(r) - t\right)\, dr + F_1(t)\}}{r}\, dr + F_2(t) \tag{5.15}$$

where $F_1(t)$, $F_2(t)$ have to be determined by the boundary conditions on ϕ_1. Finally one can obtain also an expression for the solution for η in terms of quadratures.

For example if the initial perturbation in ρ is $\rho_1(0,r) = ar$ where a is a constant then

$$F(x) = a\sin\left(\sqrt{\frac{c}{2}}x\right)$$

and

$$\rho_1(t,r) = a\sin\left(\arcsin(r) - \sqrt{\frac{c}{2}}t\right) = a\left\{r\cos\left(\sqrt{\frac{c}{2}}\,t\right) - \sqrt{1-r^2}\sin\left(\sqrt{\frac{c}{2}}\,t\right)\right\}. \quad (5.16)$$

The evaluation of ϕ_1 using the second equation in (5.13) and (5.16) is straightforward. It should be obvious how one can generalize this example to other expressions for $\rho_1(0,r)$.

A second approach to the solution of the system (5.13) is to assume exponential dependence in time, viz.

$$\rho_1 = e^{\alpha t}R(r), \;\; \eta_1 = e^{\alpha t}E(r), \;\; \phi_1 = e^{\alpha t}P(r) \quad (5.17)$$

This ansatz reduces (5.13) to a system of ordinary differential equations

$$\begin{aligned} \frac{\sqrt{2c(1-r^2)}}{2}R(r)' + \alpha R(r) = 0, \\ P(r)'' + rP(r)' - crR(r) = 0, \\ \frac{\sqrt{2c(1-r^2)}}{2}E(r)' + \alpha E(r) + P(r) = 0. \end{aligned} \quad (5.18)$$

As before the equation for $R(r)$ can be solved analytically,

$$R(r) = C_1 \exp\left(-\alpha\sqrt{\frac{2}{c}}\arcsin(r)\right)$$

while the equations for $P(r)$ and $E(r)$ can be solved by quadratures or numerically.

A numerical approach to the solution for ϕ_1 and η_1 in (5.13) is also possible.

5.3 Perturbations from the steady state $\eta_0 = 1$

As in the previous subsection we consider again a disk of radius 1 and let $S(\rho) = \alpha\rho$. The general steady state solution for ρ is given by (5.9). Assuming no singularities in ρ (ie. no

protostar at the origin) we must set $C_2 = 0$ in this equation. Furthermore since $\rho \geq 0$ it follows that we must have $\sqrt{-c/\alpha} = \beta$ where β is the first zero of J_0. Thus

$$\rho_0 = J_0(\beta r), \;\; \phi_0 = \alpha\rho_0$$

(where we normalized rho_0 at $r = 0$ to be 1).

For a perturbation from this steady state in the form given by (5.12) we obtain to first order in ϵ the following system of equations:

$$(\phi_1)_{rr} + \frac{1}{r}(\phi_1)_r - c\rho_1 = 0, \tag{5.19}$$
$$(\eta_1)_t - \alpha\rho_1 + \phi_1 = 0$$
$$(\rho_1)_t - \beta J_1(\beta r)(\eta_1)_r = 0$$

Where J_1 is Bessel function of the first kind of order 1. The evolution of an initial perturbation $\rho_1 = exp(-5r)$ with $\alpha = 0.01$ from the steady state is plotted in $Fig5$. This figure shows that as time progresses there is an accumulation of matter near the center of the disk. At the same time there is an initial separation between the core and the rest of the disk.

We computed also the solution to the system (5.5) with an initial matter distribution $\rho(0, r) = \frac{1+\sin(2\pi r)}{2}$, $c = 0.01$ and $t \in [0, 9]$. The results of the simulation (Fig. 6) show that as time progresses matter is starting to build up in the vicinity of the center of the disk and around $r = 1$. At the same time there is a decrease in matter density in between these two points.

6 Summary and Conclusions

In previous publications [18-20] we treated only the steady states of two dimensional self gravitating fluid. In this paper we generalized this model to include disk rotation and assessed the impact of this addition on the distribution of matter in the disk. We were able

also to address the time dependent evolution of this fluid under restrictions on its vorticity. This enabled us to simplify considerably the equations which govern its evolution. While this is a highly idealized model in the context of astrophysical applications it may still provide some analytical insights for more elaborate models.

In this paper we considered only radial solutions of this model. More general solutions which are not radial will have to be explored next.

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List of Captions

Fig. 1 Steady states with $\alpha = -19.4$, $c = 1$ and boudary conditions $\rho(0) = 1$, $\rho(8) = 0$ with different values of ω

Fig. 2 Steady states with $\alpha = -19.4$, $c = 1$ and boudary conditions $\rho(0) = 0.35$, $\rho(8) = 0.25$ with different values of ω

Fig. 3 The steady state that corresponds to (5.10)

Fig. 4 The steady state that corresponds to (5.11)

Fig. 5 Using (5.19) to solve for ρ_1 with $\alpha = -0.01$, $c = 0.0578$ and initial perturbation $\rho_1 = exp(-5r)$ with a protostar at the origin.

Fig. 6 Using (5.5) to compute the evolution of ρ. The initial matter distribution is $\rho(0, r) = \frac{1+\sin(2\pi r)}{2}$ and $c = 0.01$. No protostar at the origin.

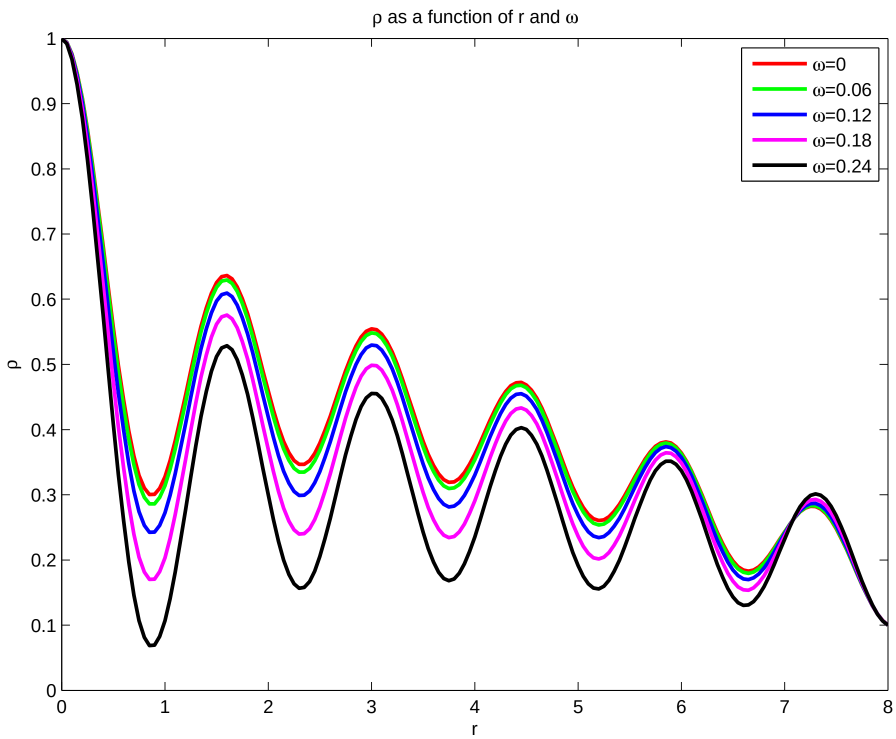

ρ as a function of r and ω
ω=0
ω=0.06
ω=0.12
ω=0.18
ω=0.24
ρ
r
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7
8

Figure 1:

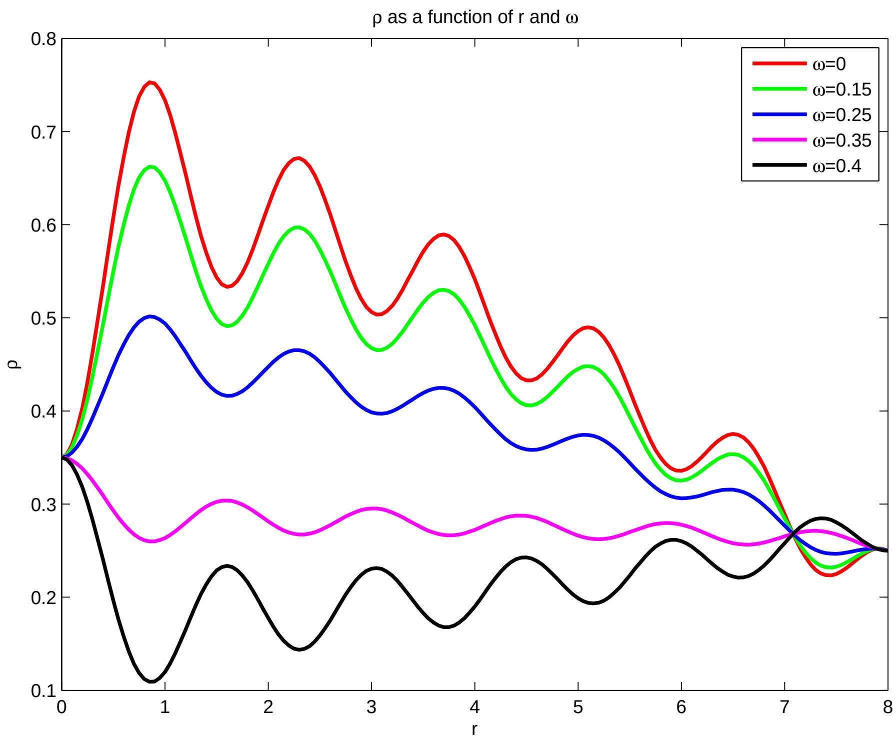

ρ as a function of r and ω
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
ρ
0
1
2
3
4
5
6
7
8
r
ω=0
ω=0.15
ω=0.25
ω=0.35
ω=0.4

Figure 2:

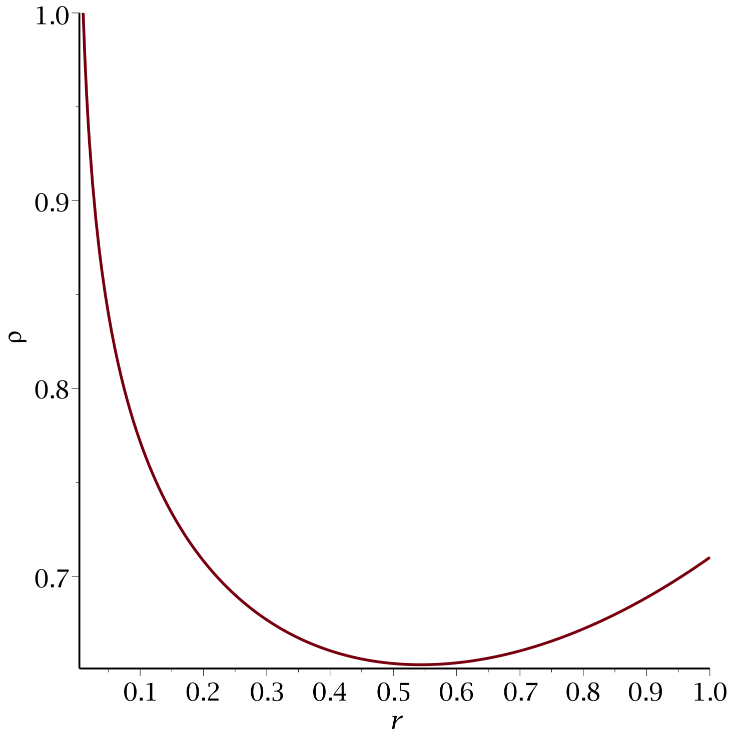


Figure 3:

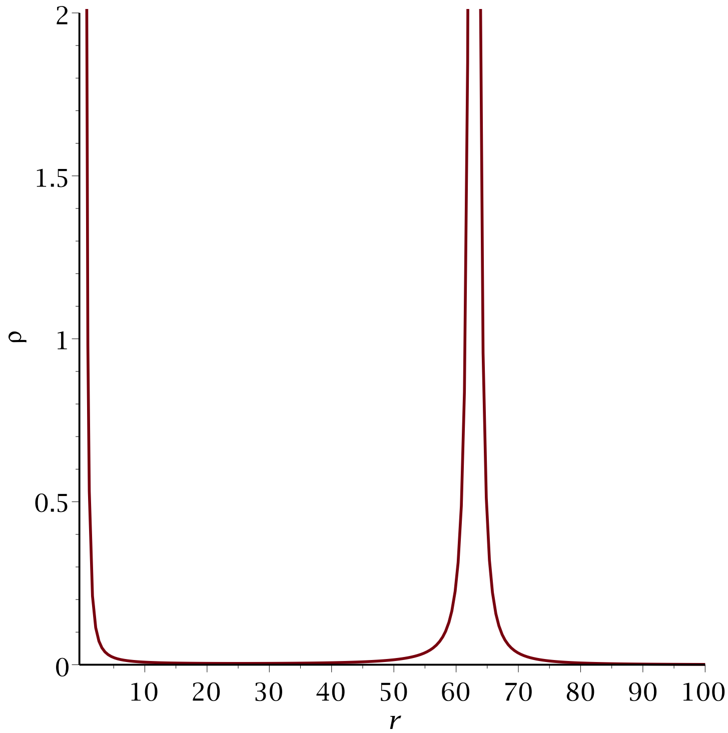
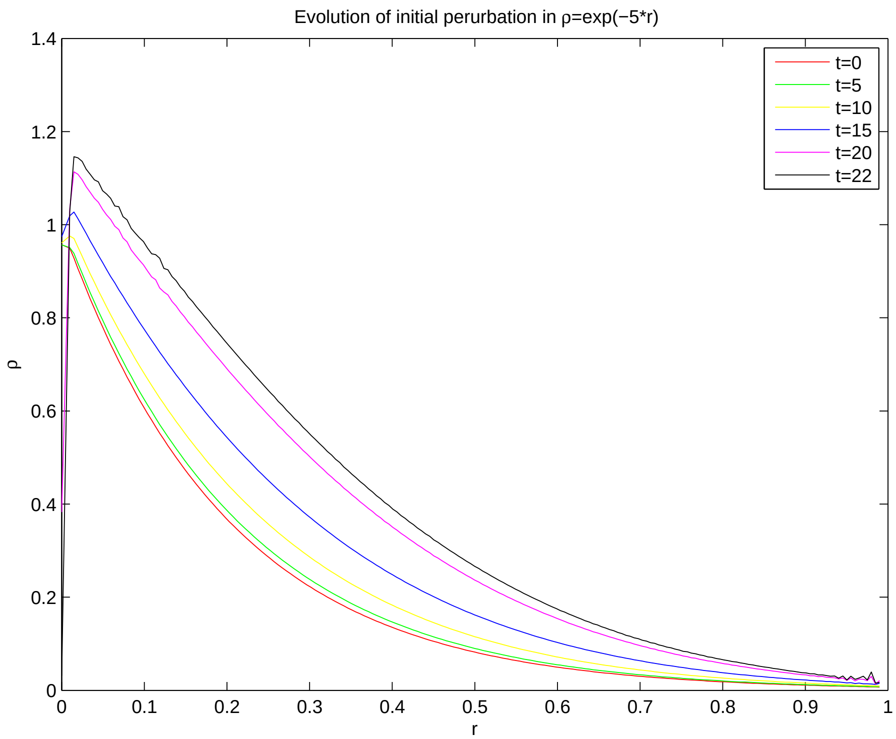


Figure 4:



Evolution of initial perurbation in ρ=exp(−5*r)
ρ
r
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.1
0.3
0.5
0.7
0.9
t=0
t=5
t=10
t=15
t=20
t=22

Figure 5:

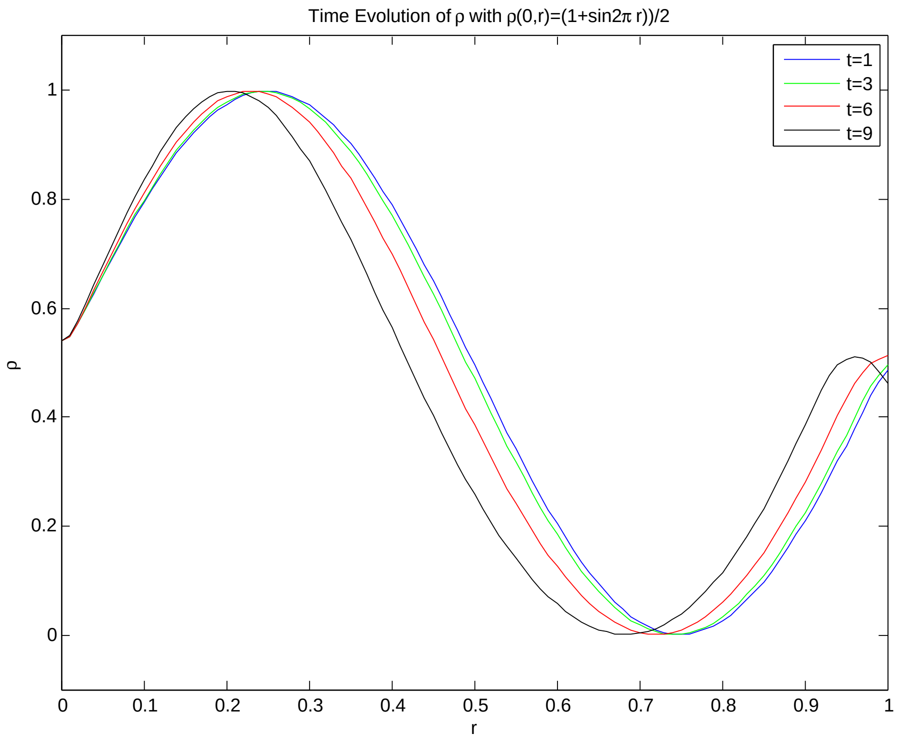

Time Evolution of ρ with ρ(0,r)=(1+sin2π r))/2
t=1
t=3
t=6
t=9
ρ
r
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 6: